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A NEW APPROACH TO DETERMINATION OF THE MOST CRITICAL MULTI-STATE COMPONENTS IN MULTI-STATE SYSTEMS

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A new approach to determination of the most critical multi-state components in multi-state systems is presented. The approach is based on solving an appropriate reliability optimization problem. The consideration is restricted on coherent and homogenous systems. Multi-state components may have different and distinct states which vary from complete failure to perfect functionality. The number of possible states is fixed and the same for all components and the system as a whole. The states correspond to different performance levels of components and the system. It is supposed that for each component state the corresponding probability and cost is known and that a higher state implies a higher cost. Further on, it is supposed that states of components are mutual statistically independent random variables. An original mathematical model for reliability calculation is developed and a corresponding optimization problem for identifying the most critical components is formulated and solved on numerical example.

Key words: Multi-state systems, Importance measures, Optimization

INTRODUCTION

Reliability Centered Maintenance (RCM) is one of the effective approach for determining the maintenance programs in practice. This approach tends to identify the components that are critical for the system reliability and to direct maintenance efforts towards these components [17]. Therefore, it is important to understand the physics of failure of a system and to understand the impact of component reliability on system reliability. Starting from Birnbaum [02] and Vesely [13], a lot of different approaches to treating component importance, called importance measures, have been developed and reported [05]. Most of these approaches rank the system components according to calculated values of a given importance measure. The components with higher rank are then considered the most critical. However, they rank only individual components' influence and do not consider the influence of combinations or groups of components on system reliability [18].

The majority of importance measures refer to systems and their components with two (binary) states: operational state and failed state. However, different components' degradation states between perfect functionality and complete failure can appear in many systems. Such components are called multi state components. Those components' degraded states can cause system's degraded but still working state [14]. Such systems are called multi state systems and they can function with various levels of efficiency, commonly called performance levels. Hence, systems with binary states can be considered the special cases of multi state systems.

According to Yingkui and Jing [15], multi-state component criticality and importance analysis is one of the most promising research direction in the field of multi state systems. Although the analysis of multi state systems was introduced since the seventies [01], the interest for importance measures of such system started at the end of the 20th century [16]. Levitin G, Lisnianski A [07] applied traditional importance measures for a system with two performance levels and multi-state components. Ramirez-Marquez et al [12] analysed two types of multi state components criticality: the impact of a component as a whole on system reliability and the impact of a particular component state on system reliability. Later, Ramirez-Marquez and Coit [11] proposed composite importance measure for multi state systems that includes costs constraint in components criticality determination. Zaitseva [16] and Kvassay, Zaitseva, and Levashenko [06] used logical differential calculus for importance analysis of multi-state systems.

In this paper, we investigate the problem of determining the most critical multi state components in multi state systems. The main idea is to reduce multi state to binary problem. This idea was already used by Levitin, Podofillini, and Zio [08]. They introduced a performance threshold for the system state and divided system's states in two disjoint subsets consisting of the states above and below the threshold level. Consequently, they re-introduced binary logic into problem of determining components' importance in the multi state systems.

In addition, we propose an approach for simultaneous determination of a group of the most critical components. The approach is based on solving an optimization problem that is based on the well-known set covering problem [03]. The remainder of the paper is organized as follows: Next section is devoted to multi state components and systems and calculation of given state probabilities for series and parallel systems. New approach for determining the set of the most critical multi state system

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components is introduced afterwards. Numerical results and discussion are presented in the last section.

MULTI STATE SYSTEMS

Suppose that system have *n* components and that each of its components can be in one of the *m* states $S=\{0,1,...,m-1\}$, where state 0 corresponds to the complete component's failure and the state *m-1* corresponds to the perfect functionality. In order to introduce the multi states system and its probabilities, we use the following notation.

$I=\{1,2,...,n\}$ - set of components,

$S_i=\{0,1,...,m_i-1\}$ - set of states of the component *i*,

$S=\{0,1,...,m-1\}$ - set of system states,

$p_{ij}=\mathrm{P}(k_i=j)$ - probability associated with *j-th* state of the *i-th* component, $i\in I,\ j\in S$,

$A_i^{=j}=\mathrm{P}(k_i=j)$ - probability that the i-th component is in the state *j*, $j\in S$,

$A_i^{\geq j}=\mathrm{P}_i(k_i\geq j)=\sum_{k=j}^{m-1}p_{ik}$ - probability that the *i-th* component is in the state *j* or higher, $i\in I,\ j\in S$,

$A_i^{\leq j}=\mathrm{P}_i(k_i\leq j)=\sum_{k=0}^{j}p_{ik}$ - probability that the *i-th* component is in the state *j* or lower, $i\in I,\ j\in S$,

$A^{=j}=\mathrm{P}(s=j)$ - probability that the system is in the state *j*, $j\in S$,

$A^{\geq j}=\mathrm{P}(s\geq j)$ - probability that the system is in the state *j*, or higher, $j\in S$,

$A^{\leq j}=\mathrm{P}(s\leq j)$ - probability that the system is in the state *j*, or lower, $j\in S$,

The first problem in reliability analysis of multi state system is to determine the structure function $\phi_i(k)=\phi_i(k_1,k_2,...,k_n):S_1\times S_2\times...\times S_n\to S$.Then, probabnility for each possible state of the system should be determined on the basis of state probabnilities for each component.

The set of components' states is finite. Therefore:

$$\sum_{j=0}^{m-1}p_{ij}=1 \qquad 1)$$

Since the higher component's state number correspond to the higher component's performance level, it holds that:

$$A_i^{\geq 0}=\mathrm{P}_i(k_i\geq 0)=\sum_{j=0}^{m-1}\mathrm{P}(k_i=j)=\sum_{k_i=0}^{m-1}p_{ik_i}=1,\ i\in I, \qquad 2)$$

$$A_i^{\leq m-1}=\mathrm{P}_i(k_i\leq m-1)=\sum_{j=0}^{m-1}\mathrm{P}(k_i=j)=\sum_{k_i=0}^{m-1}p_{ik_i}=1,\ i\in I, \qquad 3)$$

$$A_i^{\geq m-1}=\mathrm{P}_i(k_i\geq m-1)=\sum_{k_i=m-1}^{m-1}p_{ik_i}=p_{im-1},\ i\in I, \qquad 4)$$

$$A_i^{\leq 0}=\mathrm{P}_i(k_i\leq 0)=\sum_{k_i=0}^{0}p_{ik_i}=p_{i0},\ i\in I, \qquad 5)$$

$$A_i^{=j}=A_i^{\geq j}-A_i^{\geq j+1},\ i\in I,\ j\in S\setminus m-1, \qquad 6)$$

$$A_i^{=j}=A_i^{\leq j}-A_i^{\leq j-1},\ i\in I,\ j\in S\setminus 0. \qquad 7)$$

Analogously, the higher system's state number correspond to the higher system's performance level, and:

$$A^{\geq 0}=\mathrm{P}(s\geq 0)=\sum_{j=0}^{m-1}\mathrm{P}(s=j)=\sum_{j=0}^{m-1}A^{=j}=1, \qquad 8)$$

$$A^{\leq m-1}=\mathrm{P}(s\leq m-1)=\sum_{j=0}^{m-1}\mathrm{P}(s=j)=\sum_{j=0}^{m-1}A^{=j}=1, \qquad 9)$$

$$A^{\geq m-1}=\mathrm{P}(s\geq m-1)=\sum_{s=m-1}^{m-1}\mathrm{P}(s=m-1)=\sum_{s=m-1}^{m-1}A^{=m-1}=A^{=m-1}, \qquad 10)$$

$$A^{\leq 0}=\mathrm{P}(s\leq 0)=\sum_{s=0}^{0}\mathrm{P}(s=0)=\sum_{s=0}^{0}A^{=0}=A^{=0}, \qquad 11)$$

$$A^{=j}=A^{\geq j}-A^{\geq j+1},\ j\in S\setminus m-1, \qquad 12)$$

$$A^{=j}=A^{\leq j}-A^{\leq j-1},\ j\in S\setminus 0. \qquad 13)$$

HOMOGENOUS SYSTEM

The consideration in this paper is restricted on homogenous systems, i.e. on systems in which the number of states for each component and for the system is the same $m_1=...=m_n=m$.

Further on, the treshold concept is used for reliability calculation.

Treshold concept

The state *r* is called the component's threshold state if the component is considered in the failure when it is in the state lower than *r*. In other words, a component is operational if it is in the state *r* or higher.

The state *r* is called the system's threshold state if the system is in the failure when it is in the state lower than *r*. This means that the system is operational if it is in the state *r* or higher.

Using the threshold approach, appropriate definitions of system structure and reliability functions may be introduced and some kind of binary logic applied.

Series system

A series system is in failure state if any of its components is in failure. It is as weak as its weakest component. In other words, the system is operational if all its components are operational.

Series system is in the state *j* if no one of its components is in the state lower than *j* and at least one is in the state *j*:

$$j=\min\{k_i,i\in I\} \qquad 14)$$

The probability that the system is in state j or higher is:

$$A^{\geq j} = \prod_{i \in I} \mathrm{P}(k_i \geq j) = \prod_{i \in I} A_i^{\geq j} \quad \text{and} \quad A^{=j} = A^{\geq j} - A^{\geq j+1}$$

Based on (8), (10) and (12), it holds that $\sum_{j=0}^{m-1} A^{=j} = 1$.

Proof: $\sum_{j \in S} A^{=j} = A^{=0} + A^{=1} + A^{=2} + \ldots + A^{=m-2} + A^{=m-1}$

Based on (12):

$$\sum_{j \in S} A^{=j} = (A^{\geq 0} - A^{\geq 1}) + (A^{\geq 1} - A^{\geq 2}) + \ldots + (A^{\geq m-2} - A^{\geq m-1}) + A^{=m-1}$$

and based on (10):

$$\sum_{j \in S} A^{=j} = (A^{\geq 0} - A^{\geq 1}) + (A^{\geq 1} - A^{\geq 2}) + \ldots + (A^{\geq m-2} - A^{\geq m-1}) + A^{\geq m-1} = A^{\geq 0}$$

Finally, according to (8): $\sum_{j \in S} A^{=j} = 1$.

Parallel system

A parallel system is operational if any of its components is operational. In other words, the system is in failure if all its components are in failure.

Parallel system is in the state *j* if no one of its components is in the state lower than *j* and at least one is in the state *j*:

$$j = \max\{k_i, i \in I\} \tag{15}$$

The probability that the system is in state j or higher is:

$$A^{\geq j} = \prod_{i \in I} \mathrm{P}(k_i \geq j) = \prod_{i \in I} A_i^{\geq j} \quad \text{and} \quad A^{=j} = A^{\geq j} - A^{\geq j+1} \tag{16}$$

This condition can also be expressed as:

$$A^{\leq j} = \prod_{i \in I} \mathrm{P}(k_i \leq j) = \prod_{i \in I} A_i^{\leq j} \quad \text{and} \quad A^{=j} = A^{\leq j} - A^{\leq j-1} \tag{17}$$

Based on (9), (11) and (13), it holds that $\sum_{j=0}^{m-1} A^{=j} = 1$.

Proof: $\sum_{j \in S} A^{=j} = A^{=0} + A^{=1} + A^{=2} + \ldots + A^{=m-2} + A^{=m-1}$

Based on (13):

$$\sum_{j \in S} A^{=j} = A^{=0} + (A^{\leq 1} - A^{\leq 0}) + (A^{\leq 2} - A^{\leq 1}) + \ldots + (A^{\leq m-2} - A^{\leq m-3}) + (A^{\leq m-1} - A^{\leq m-2})$$

and based on (11):

$$\sum_{j \in S} A^{=j} = A^{\leq 0} + (A^{\leq 1} - A^{\leq 0}) + (A^{\leq 2} - A^{\leq 1}) + \ldots + (A^{\leq m-2} - A^{\leq m-3}) + (A^{\leq m-1} - A^{\leq m-2}) = A^{\leq m-1}$$

Finally, according to (9): $\sum_{j \in S} A^{=j} = 1$.

Coherent system

Definitions of minimal cut set and minimal path set are used in analysing coherent systems of general configuration. For binary state systems the definitions are following.

Cut set of a binary system is a set of components whose failures cause the system failure.

Minimal cut set (MCS) is a cut set reduced to the minimum number of components whose failures cause the system failure.

MCS can be expressed as a parallel system of its components, i.e. if any of the components is operational then the set is no longer cut set.

A coherent system can be expressed as a serial system of its MCSs, i.e. that the system failes if at least one of its MCSs occurs.

Using the concept of treshold state, the following similar definitions are used.

MCS in a multi state svstem may be in one of the *m* possible states $S = \{0, 1, \ldots, m-1\}$.

MCS is in the state *j* or lower if all its components are in the state *j* or lawer.

The system is in the state *j* or higher if all of its MCSs are in the state *j* or higher.

PROBLEM STATEMENT AND MATHEMATICAL FORMULATION

It is supposed that the system threshold *r* and MCSs of the sistem are given and the task is to find the set of the most critical components.

Let *C* be a set of p MCSs of a given multi state system, $C = \{C_1, \ldots, C_n\}$. The state s_l of the *l*-th MCS is [10]:

$$s_l = \max\{k_i, i \in C_l\} \tag{18}$$

while the state of the system is:

$$s = \min_{C_l \in C}\{s_l\} = \min_{C_l \in C}\{\max\{k_i, i \in C_l\}\} \tag{19}$$

Recall that the system is in failure if at least one of its MCSs is in state lawer than *r*, i.e. the system is operational if all its MCSs are in state *r* or higher. Further on, a MCS is in state *r* or higher if at least one of its components is in the state *r* or higher. The problem is to find minimal set of components whose states *r* or higher enable that the system is in state *r* or higher. Components in that set are the the most critical multi-state components in the multi-state system.

Suppose that all the system's components are in states which are lower than treshold *r*. According to the equation (18), if some of the components $i \in C_l$ increases its performances and achieves the state *r* the corresponding MCS C_l will be in state *r* and is no more cut set.

Therefore, such MCS will be eliminated as a cause of the system failure. Moreover, all MCSs $C_l \in C$ containing the component *i* will be in the state *r* and eliminated for the same reason. If all MCSs $C_l \in C$ are in the state *r* or higher, the system will be operational, i.e. in state *r* or higher.

The problem is to find *K* components whose achieving the state *r* or higher maximizes the total number of eliminated MCSs. A particular form of the defined problem can be formulated as follows: find the minimal *K* such that all MCSs are eliminated.

Let x_i be the binary variable equal to 1 when the component *i* is in the state *r*, and 0 otherwise, $i \in I = \{1, 2, ..., n\}$.

The problem of finding the minimal number of components whose state *r* will cause elimination of all MCSs is the folowing optimization problem:

$$\min \sum_{i \in I} x_i \quad 20)$$

s.t.

$$\sum_{i \in C_l} x_i \geq 1, \; C_l \in C \quad 21)$$

$$x_i \in \{0,1\}, \; i \in I \quad 22)$$

Objective function represents the total number of components which have to be in state *r* in order the system be operational. The constrainta (21) are related to the requirement that all MCSs must be eliminated, i.e. that each MCS must contain at least one component in the state *r*.

Note: The above formulated problem is equivalent to the problem of determining the shortest minimal path set (MPS).

NUMERICAL EXAMPLE

The proposed approach will be illustrated on the multi state system represented in Figure 1.

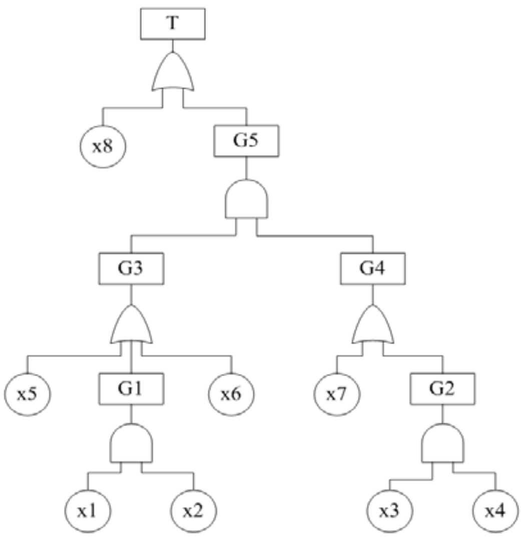

Figure 1: Fault tree of the system

The root node T corresponds to the top event and represents the system failure. The leaves x1-x8 are basic events. It is supposed that the system and its components may have three states:

0 - failure state,

1 - degraded but still working state, and

2 - perfect functionality state, respectively.

Let the system and all eight components have three states: 0, 1 and 2, that represent failure, degraded but still working state and perfect functionality, respectively.

Structure function is given as a disjunctive normal form where each conjunction represents one MCS [04,09]. For the fault tree in Figure 1, MCSs are: $C_1=\{x_8\}$, $C_2=\{x_5, x7\}$, $C_3=\{x_6, x_7\}$, $C_4=\{x_3, x_4, x_5\}$, $C_5=\{x_1, x_2, x_7\}$, $C_6=\{x_3, x_4, x_6\}$ and $C_7=\{x_1, x_2, x_3, x_4\}$.

Using the equation (19) and the set MCSs, the system state is:

$$s=\min\{s_1,...,s_7\}=\min\{\max\{k_8\},\max\{k_5, k_7\},\max\{k_6, k_7\},\max\{k_3, k_4, k_5\},\max\{k_1, k_2, k_7\},\max\{k_3, k_4, k_6\},\max\{k_1, k_2, k_3, k_4\}\}$$

Assuming that the probabilties of all basic events are equal, Birnbaum importance measure [02] and Fussell–Vesely importance measure [13] give the same components ranking: x_8, x_7, x_5, x_6, x_3, x_4, x_1 and x_2, i.e. x_8 is the most important and x_2 the least important component.

According to the optimal solution of the mathematical model (20-22), there are two sets containing three components: x_8, x_7 and x_3, or x_8, x_7 and x_4 . These sets are considered as the minimal set of the most important components.

$$s=\min\{\max\{1\},\max\{0,1\},\max\{0,1\},\max\{1,0,0\},\max\{0,0,1\},\max\{1,0,0\},\max\{0,0,1,0\}\}=1$$

However, if the states of the three first ranked components obtained by Birnbaum and Fussell–Vesely importance measures (x_8, x_7 and x_5) is increased to 1, the state of the system is:

$$s=\min\{\max\{1\},\max\{1,1\},\max\{0,1\},\max\{0,0,1\},\max\{0,0,1\},\max\{0,0,0\},\max\{0,0,0,0\}\}=0$$

This is the consequence of the fact that traditional importance measures do not take sufficiently into account the interdependence of components and are not able to simultaneously isolate the most important group of components.

CONCLUDING REMARKS

The problem of determining the set of the most critical components of the multi state system with multi state components was considered. This set is defined as the set of components whose operational states imply that all MCSs are no more cut sets. In other words, there exist no one MCS so that all its components are different from components from the set of the most critical components. At least one component from any MCS must be in the set of the most critical components.

Finding the set of most critical components was formulated as a well-known set covering problem. It was solved using available software for integer linear programming problems. It is interesting to point out that obtained solution on an illustrative example differs from the ones obtained by the Birnbaum and Fussell-Vesely importance measures. It was not unexpected because Birnbaum and Fussell-Vesely importance measures are determined considering each component individually. The proposed approach takes into account the group influence an consequently the group importance measure. Further research should include different probabilities of performance levels that correspond to the system's and components' states as well as the costs of achieving given states.

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